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Asymptotic Integration of Delay Differential Systems

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INTRODUCTION

In this paper we consider the asymptotic integration of a delay differential equation. This means that we are dealing with non-autonomous equations which are asymptotically autonomous, and we look for formulae for the solutions at large values of the independent variable.

As a well-known example, we can mention

$$\frac{dx}{dt} = p(t) \cdot (x(t) - x(t-\tau)), \tag{1}$$

where p is in L^2 or $\int_{t-\tau}^{t} |p(s)|\, ds \leqslant k < 1$. Asymptotically, (1) is like $\dot{x} = 0$, and indeed it has been proved in [1, 4] that the solutions of (1) are asymptotically constant. Another example is

$$\frac{dx}{dt} = -a\, x(t - r(t)), \tag{2}$$

where $r(t) \to 0$, $t \to \infty$. The asymptotic equation is $dx/dt = -ax(t)$ and it has been proved by K. L. Cooke [5] that if r is in L^1 then the solutions of (2) are asymptotically of the form $\exp(-at) \cdot \text{Const}$.

The equation under consideration here is

$$\frac{dx}{dt} = \Lambda x(t) + L(t, x_t), \qquad \text{for } t \geqslant t_0, \tag{3}$$

where $x(t) \in \mathbb{R}^n$, x_t denotes, as usual, the function defined on $[-\tau, 0]$ by $x_t(s) = x(t+s)$, $-\tau \leqslant s \leqslant 0$ [10]. Here τ is the maximum delay in (3). We will state now the assumptions on Λ and L.

(H_1) Λ is linear from $\mathbb{R}^n$ into $\mathbb{R}^n$, represented by a diagonal matrix with n distinct entries $(\lambda_i)_{i=1,\ldots,n}$;

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(H_2) For each t, $L(t, \cdot)$ is linear continuous from $C([-\tau, 0], \mathbb{R}^n)$ into $\mathbb{R}^n$; $t \to L(t, \cdot)$ is continuous and $\|L(t, \cdot)\|$ is in $L^2(t_0, +\infty)$.

The problem of asymptotic integration of such equations has been studied by J. R. Haddock and R. Sacker [9] for a more particular model equation

$$\frac{dx}{dt} = (\Lambda + A(t))\, x(t) + B(t)\, x(t-\tau), \tag{4}$$

with A and B in $L^2(t_0, +\infty)$.

Haddock and Sacker proposed a study of (4) when trying to extend previous results by Hartman [12], Hartman and Wintner [13], Atkinson [3], and Harris and Lutz [11], notably for ordinary differential equations of the form

$$\frac{dx}{dt} = (\Lambda + A(t)) \cdot x(t), \qquad \text{where } A \text{ is in } L^2. \tag{5}$$

In [9], a first result was established for the scalar case. In view of this result, J. Haddock and R. Sacker conjectured an asymptotic formula for the vectorial case. The conjecture states that there exists a matrix function F, $F(t) \to 0$ as $t \to +\infty$, and for each solution x, a constant vector c and a function f, $f(t) \to 0$ as $t \to +\infty$ such that

$$x(t) = [Id + F(t)] \cdot \exp\left(\int_0^t \mathcal{A}(s)\, ds\right) \cdot [c + f(t)], \tag{6}$$

where

$$\mathcal{A}(t) = \Lambda + \operatorname{diag}\{A(t)\} + \operatorname{diag}\{B(t)\} \cdot e^{-\tau\Lambda}.$$

We may observe that such problems were investigated earlier in a different perspective by J. Ryabov [16], R. Driver [6], and I. Györi [7, 8].

In this paper we mainly prove a result very close to the Haddock and Sacker conjecture for Eq. (3) under more general assumptions than in [9]. Our formula differs from (6) essentially in that $F(t)$ is replaced by a functional $G(t)$ defined on the space $C([-2\tau, t], \mathbb{R}^n)$.

Of course, we still have $G(t) \to 0$ as $t \to +\infty$. Our result can be described as follows: for each solution x of (3), there exist a constant vector c, a function η_1 with values in $\mathbb{R}^n$, $\eta_1(t) \to 0$ as $t \to +\infty$, and a function η_2, $\eta_2(t) \in C([-2\tau, t], \mathbb{R}^n)$, $\eta_2(t) \to 0$ as $t \to +\infty$ such that

$$x(t) = \exp\left\{\int_0^t \mathcal{A}(s)\, ds\right\} \cdot [c + \eta_1(t)] + G(t)\left\{\exp \int_0^t \mathcal{A}(s)\, ds \cdot [c + \eta_2(t)]\right\}. \tag{7}$$

While this formula is generally less agreeable than (6), we have been able to establish a better one in the case that we call quasi-triangular. In this situation, the formula (7) holds with $G(t)=0$. This covers the scalar case and yields then the same result as in [9].

The method used in our paper is very close to the techniques employed in the study of partial stability. A few fundamental results along these lines are collected in Section 1. Section 2 deals with the quasi-triangular systems and Section 3 discusses the general situation. Our treatment relies on changes of variables which in particular allow us to give an inductive proof of the result with respect to the dimension n.

1. Fundamental Results

Proposition 1. *Consider the equation*

$$\frac{dx}{dt}=B(t,x_t)+P(t,x_t), \tag{8}$$

in which

$$(\mathrm{H}_3)\begin{cases} B(t,c)=0 \text{ (i.e., } B \text{ is a balanced term as defined in } [14]); \\ |B(t,\varphi)-B(t,\psi)|\leqslant\int_\tau^0 b(t,s)\left|\frac{d\varphi}{ds}-\frac{d\psi}{ds}\right|ds, \\ \text{for every } \varphi, \psi \text{ absolutely continuous, and} \\ \limsup_{t\to\infty}\int_\tau^0 b(t-s,s)\,ds<1, \end{cases}$$

$$(\mathrm{H}_4)\begin{cases} |P(t,\varphi)-P(t,\psi)|\leqslant p(t)\cdot|\varphi-\psi|_{C([-\tau,0],\mathbb{R}^n)}, \\ \text{with } p(t) \text{ and } P(t,0) \text{ in } L^1. \end{cases}$$

Then for each solution x of (8), $\lim_{t\to\infty}x(t)$ *exists. Moreover, for t_0 large enough, and each c in $\mathbb{R}^n$, there exists a solution of* (8) *defined on* $[t_0-\tau,+\infty)$, *with c as a limit at infinity.*

Remark 1. The functional: $\varphi\to\lim_{t\to+\infty}x(t_0,\varphi)(t)$ (with usual notations) is continuous for each t_0; and Proposition 1 states that it is surjective if t_0 is large enough. In fact, as $t_0\to+\infty$, it tends to the Dirac function $\varphi\to\varphi(0)$.

This proposition was proved in [1]. The convergence was also established in [4]. In this last paper, a significantly more general condition than the one in Proposition 1 was obtained by coupling in some way $p(t)$ and $h(t, s)$ (see also [1], for a discussion on such conditions).

PROPOSITION 2. *Consider the equation*

$$\frac{dx}{dt} = M(t, x_t) + N(t, y_t)$$
$$\frac{dy}{dt} = P(t, x_t) + Q(t, y_t), \tag{9}$$

with $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$;

$$N(\text{resp. } P)\colon [t_0, +\infty) \times C([-\tau, 0], \mathbb{R}^n) \to R^n(\text{resp. } \mathbb{R}^m),$$
$$N(\text{resp. } Q)\colon [t_0, +\infty) \times C([-\tau, 0], \mathbb{R}^m) \to \mathbb{R}^n(\text{resp. } \mathbb{R}^m),$$

where M, N, P, Q *are continuous linear functionals with respect to the second variable. Assume moreover that*

(H_5) *the equation*

$$\frac{dx}{dt} = M(t, x_t) \tag{10}$$

is stable;

(H_6) *the equation*

$$\frac{dy}{dt} = Q(t, y_t) \tag{11}$$

is exponentially stable;

(H_7) $\|N(t, \cdot)\|$ *and* $\|P(t, \cdot)\|$ *are in* L^2.

Let (x, y) *be a solution of* (9). *Then* x *is bounded and* $\lim_{t \to \infty} y(t) = 0$. *Moreover, if for all the solutions* $u(t)$ *of* (10), $\lim_{t \to \infty} u(t)$ *exists, the same holds with the solutions of* (9).

Remark 2. (a) We will not need Proposition 2 in its full generality but this result is of independent interest.

(b) We need to assume linearity of M and Q because we will use variations of constant formulas from Eqs. (10) and (11). But, a Lipschitzean nonlinearity in N and P is admissible.

Proof of Proposition 2

Notation 1. We will denote by $U(t, s)$ (resp. $V(t, s)$), $t \geqslant s$, the solution operator associated with (10) (resp. (11)).

So $U(t, s) \cdot x_s = x_t$ for each solution x of (10), and $V(t, s) . y_s = y_t$ for each solution y of (11). From (H_5) and (H_6) (in Proposition 2), we deduce

$$|U(t, s)| \leqslant K, \qquad |V(t, s)| \leqslant K \cdot \exp[-\alpha(t-s), \qquad t \geqslant s,$$
$$\text{for some } \alpha > 0,\ K < +\infty\ [10]. \tag{12}$$

We will express the assumption (H_7) in the form

$$|N(t, \varphi)| \leqslant n(t) \cdot |\varphi|_{\infty}; \qquad |P(t, \psi)| \leqslant p(t) \cdot |\psi|_{\infty},$$
$$\text{with } n \text{ and } p \text{ in } L^2. \tag{13}$$

Let (x, y) be a solution of (9). Using the variation of constants formula for (10) and (11), we can represent x_t, y_t as

$$\begin{aligned} x_t &= U(t, t_0)\, x_{t_0} + \int_{t_0}^{t} U(t, s)\, X_0\, N(s, y_s)\, ds, \\ y_t &= V(t, t_0) \cdot y_{t_0} + \int_{t_0}^{t} V(t, s)\, Y_0\, P(s, x_s)\, ds, \end{aligned} \tag{14}$$

where X_0 (resp. Y_0) denotes a function with a bounded variation associated to the Dirac distribution at 0 in $\mathbb{R}^n$ (resp. $\mathbb{R}^m$). Define

Variable 1. $u(t) = |x_t|$, $v(t) = |y_t|$.

From (12) and (14) we can see immediately that (u, v) verifies a system of inequalities:

$$\begin{aligned} u(t) &\leqslant K\, u(t_0) + K \int_{t_0}^{t} n(s)\, v(s)\, ds, \\ v(t) &\leqslant K e^{-\alpha(t-t_0)}\, v(t_0) + K \int_{t_0}^{t} e^{-\alpha(t-s)}\, p(s)\, u(s)\, ds. \end{aligned} \tag{15}$$

We will show first that u is bounded and $v(t) \to 0$ as $t \to \infty$. Combining the two inequalities of (15) we can derive an inequality involving only u (resp. only v). This gives

$$\begin{aligned} u(t) \leqslant K\, u(t_0) &+ K^2 \left(\int_{t_0}^{t} n(s)\, e^{-\alpha(s-t_0)}\, ds \right) \cdot v(t_0) \\ &+ K^2 \cdot \int_{t_0}^{t} n(s) \cdot \left(\int_{t_0}^{s} e^{-\alpha(s-\tau)}\, p(\tau)\, u(\tau)\, d\tau \right) ds. \end{aligned} \tag{16}$$

Notation 2. Let

$$b(t_0, u_0, v_0) = Ku_0 + K^2 \left(\int_{t_0}^{\infty} n(s)\, e^{-\alpha(s-t_0)}\, ds \right) \cdot v_0, \qquad \text{for } u_0, v_0 \geqslant 0,$$

$$c(t_0) = K^2 \cdot \int_{t_0}^{\infty} n(s) \left(\int_{t_0}^{s} e^{-\alpha(s-\tau)} p(\tau)\, d\tau \right) ds.$$

LEMMA 1. *b and c are uniformly bounded with respect to t_0. Moreover $c(t_0) \to 0$ as $t_0 \to +\infty$.*

These two assertions are easy consequences of the Schwartz inequality in L^2 and convolution product properties of L^2 by L^1 functions.

We can now continue the proof of the proposition.

In view of Notation 2 the above inequality (16) can be written as

$$u(t) \leqslant b(t_0, u(t_0), v(t_0)) + c(t_0) \cdot \max_{t_0 \leqslant s \leqslant t} u(s), \tag{17}$$

which implies the same with $u(t)$ changed into $\max_{t_0 \leqslant s \leqslant t} u(s)$ on the left hand side. So, if t_0 is large enough, $c(t_0)$ will be <1 (from Lemma 1), and then we get

$$u(t) \leqslant \frac{b(t_0, u(t_0), v(t_0))}{1 - c(t_0)}, \qquad t \geqslant t_0. \tag{18}$$

Using the same arguments with $+\infty$ replaced with $t_0 + T$ for T small enough (independently of t_0), we can prove that

$$\max_{s \leqslant t_0 + T} u(s) \leqslant C \cdot \max_{s \leqslant t_0} u(s),$$

with C independent of t_0 and no restriction on t_0. This means that (18) holds for all t_0, however, possibly with a different right side, and, therefore u is bounded. From the boundedness of u and $(15)_2$ we obtain readily that $v(t) \to 0$, $t \to +\infty$. This uses once more a convolution product argument and the fact that $e^{-\alpha t}$ is at the same time in L^1 and L^2. In view of variable 1, this completes the proof of the first part of Proposition 2.

Remark 3. We have more on v. For instance, v is in L^2. So y is in L^2 and, because of (13), $N(t, y_t)$ is in L^1.

We come now to the last part of Proposition 2. In this part, we assume that $U(t, s) \cdot \varphi$ converges as $t \to +\infty$.

Notation 3. $U_\infty(s) = \lim_{t \to +\infty} U(t, s) \cdot \varphi$.

We only have to look at formula (14). The first term tends to $U_\infty(t_0) \cdot x_{t_0}$: the term under the integral converges pointwise to

$U_\infty(s) X_0 \cdot N(s, y_s)$. But since $N(s, y_s)$ is in L^1 (as we noticed in Remark 3) and $U(t, s) \cdot X_0$ is bounded, the convergence is dominated. So the right side in $(14)_1$ has a limit at infinity, which means that $x(t)$ converges and we have

$$\lim_{t \to \infty} x(t) = U_\infty(t_0)\, x_{t_0} + \int_{t_0}^{\infty} U_\infty(s)\, X_0 N(s, y_s)\, ds. \tag{19}$$

We will consider now a more specific version of Proposition 2.

COROLLARY 1. *Assume* (H_6), (H_7) *and that* $|M(t, \cdot)|$ *is in* L^1.
Let (x, y) *be a solution of* (9). *Then* $\lim_{t \to \infty} x(t)$ *exists*, $\lim_{t \to \infty} y(t) = 0$. *Moreover, if* t_0 *is large enough, for each* c *in* $\mathbb{R}^n$ *there exists a solution defined on* $[t_0, +\infty)$ *such that* $\lim_{t \to +\infty} x(t) = c$.

Proof. From Proposition 1, we deduce immediately that Eq. (10) is stable and its solutions converge. So, all the conditions of Proposition 2 are satisfied, which yields the first part of Corollary 1. Moreover, $\lim_{t \to \infty} x(t)$ is given by the right side of (19).

We noticed in Remark 1 that $U_\infty(t_0) \cdot \varphi \to \varphi(0)$, as $t_0 \to +\infty$. At the same time, the integral tends to 0 (in the sense of norm on such functionals). More precisely, we have, using (15) variable 1,

$$\left| \int_{t_0}^{+\infty} U_\infty(s) \cdot x_0 N(s, y_s)\, ds \right| \leqslant \frac{K^2}{(2\alpha)^{1/2}} \left(\int_{t_0}^{+\infty} (n(s))^2\, ds \right)^{1/2} \cdot |y_{t_0}|$$
$$+ \frac{K^2 \cdot b(t_0, |x_{t_0}|, |y_{t_0}|)}{1 - c(t_0)} \cdot \frac{1}{(2\alpha)^{1/2}} \left(\int_{t_0}^{\infty} p(s)^2\, ds \right)^{1/2} \cdot \left(\int_{t_0}^{\infty} n(s)^2\, ds \right)^{1/2}.$$

So, in view of Notation 2, we obtain an estimate of the integral of the form $\varepsilon(t_0) \cdot (|x_{t_0}| + |y_{t_0}|)$, where $\varepsilon(t_0) \to 0$, as $t_0 \to +\infty$. Therefore, if we restrict our attention to constant data $x_{t_0} = x_0$ and $y_{t_0} = 0$, we will obtain

$$\lim_{t \to +\infty} x(t_0, x_0, 0)(t) = x_0 + o(x_0)$$

for t_0 large enough. This gives the desired surjectivity.

2. CASE OF A QUASI-TRIANGULAR MAP

We consider now Eq. (3). In addition to (H_1) we assume from now on that the λ_i's are ordered: for $i < j$, $\lambda_i > \lambda_j$. This assumption does not restrict the generality. We will frequently use a matrix representation for L, namely

$$L(t, \cdot) = (L_{ij}(t, \cdot))_{1 \leqslant i, j \leqslant n},$$

and, for a matrix A, we will denote by $\operatorname{diag}\{A\}$ the diagonal matrix constituted with diagonal elements of A.

THEOREM 1. *Assume* (H_1), (H_2), *and* (H_8) *hold, where*

$$(\mathrm{H}_8)\begin{cases}\textit{for some } \varepsilon>0 \textit{ and each } i,j,\ i>j, \textit{ there exists } C\geqslant 0,\\ \|L_{i,j}(t,\cdot)\|\leqslant C\cdot\exp((\lambda_i-\lambda_j-\varepsilon)\,t))).\end{cases}$$

Then for every x *solution of* (3) *there exists a constant* c *in* $\mathbb{R}^n$ *such that*

$$x(t)=\exp\left(\int^t \Lambda(s)\,ds\right)\cdot[c+o(1)],\tag{20}$$

where $\Lambda(t)=\Lambda+\operatorname{diag}\{L(t,\exp(\Lambda.))\}$.

The functional $\varphi\to\lim_{t\to\infty}\exp(-\int^t\Lambda(s)\,ds)\cdot x(t_0,\varphi)(t)$ *is continuous, and surjective if* t_0 *is large enough.*

Remark 4. (i) Obviously, this theorem applies to the triangular case, and, in particular, to the scalar case.

(ii) The result obtained here is stronger than the one conjectured in [9] since we have $F(t)=0$ (see Introduction).

The proof is based on two changes of variables which will reduce Eq. (3) to an equation of type (8) and on the application of Proposition 1 to the transformed equation.

First Change of Variables

Variable 2. $y(t)=\exp(-\Lambda t)\cdot x(t)$. The first expression for the equation on y is simply

$$\frac{dy}{dt}=\tilde{L}(t,y_t)\underset{\text{def}}{=}\exp(-\Lambda t)\cdot L(t,\exp\Lambda(t+\cdot)\cdot y_t)\tag{21}$$

where

$$\begin{aligned}\tilde{L}_{ij}(t,\cdot)&=\exp-\lambda_i t\cdot L_{ij}(t,\exp(\lambda_j(t+\cdot))\cdot(\ )_j)\\&=\exp(\lambda_j-\lambda_i)\,t\cdot L_{ij}(t,(\exp\lambda_j\cdot)\cdot(\ )_j).\end{aligned}$$

(We represent the map $s\to\exp(as)$ by $\exp(a\cdot)$). This leads to a first estimate of $\tilde{L}_{ij}$ in terms of L_{ij}:

$$\|\tilde{L}_{ij}(t,\cdot)\|\leqslant C_j\exp(\lambda_j-\lambda_i)\,t\cdot\|L_{ij}(t,\cdot)\|\tag{22}$$

with $C_j=\max_{-\tau\leqslant s\leqslant 0}\exp\lambda_j s$.

Notation 4. $\eta = \max(\{\lambda_j - \lambda_i;\, i<j\}, -\varepsilon)$ (ε as in (H_8)), $l(t) = \sum_{i,j=1}^{n} \|L_{ij}(t, \cdot)\|$.

In view of the ordering on the λ_i's and (H_2), we have $\eta<0$ and l in L^2, and in view of (22) and (H_8), we obtain new estimates for $\tilde{L}_{ij}$, namely

$$\begin{aligned} &\|\tilde{L}_{ij}(t, \cdot)\| = O(e^{\eta t}\cdot l(t)), \qquad \text{for } i\neq j \text{ and} \\ &\|\tilde{L}_{ii}(t, \cdot)\| = O(l(t)). \end{aligned} \tag{23}$$

These estimates suggest another presentation of Eq. (21), that is

$$\frac{dy}{dt} = \Delta(t, y_t) + h(t, y_t), \tag{24}$$

where $\Delta(t, \cdot) = \operatorname{diag}\{\tilde{L}(t, \cdot)\}$ is such that $\|\Delta(t, \cdot)\|$ is in L^2, and, therefore, $\|h(t, \cdot)\| = O(e^{\eta t}\cdot l(t))$ is in L^1. We rewrite Δ in the form

$$\Delta(t, \varphi) = \Delta(t, \varphi(0)) + \Delta(t, \varphi - \varphi(0)).$$

We have $\Delta(t, \varphi(0)) = \operatorname{diag}\{\tilde{L}(t, \varphi(0))\}$. So, in view of Eq. (21) and the definition of $A(t)$ given in the statement of Theorem 1,

$$\begin{aligned} \Delta(t, \varphi(0) &= \operatorname{diag}\{L(t, \exp A.)\}\cdot\varphi(0) \\ &= (A(t) - A)\cdot\varphi(0). \end{aligned}$$

Notation 5. $\tilde{A}(t) = A(t) - A$.

Notation 6. $\Delta_1(t, \varphi) = \Delta(t, \varphi - \varphi(0))$.

Δ_1 is still diagonal, but, because of the presence of $\varphi - \varphi(0)$, it can be expressed as a functional on $C([-2\tau, 0], \mathbb{R}^n)$, for $t \geqslant t_0 + \tau$, using $y_t - y(t) = \int_t^{t+} dy/ds\cdot ds$, and replacing dy/ds with the right side of (24). We get

$$|\Delta_1(t, y_t)| \leqslant \|\Delta(t, \cdot)\|\cdot\left(\int_{t-\tau}^{t} \|\Delta(s, \cdot)\|\, ds + \int_{t-\tau}^{t} \|h(s, \cdot)\|\, ds\right) \max_{[-2\tau, t]} |v(s)| \tag{25}$$

and so

$$\|\Delta_1(t, \cdot)\|_{C([-2\tau, 0],\, \mathbb{R}^n)} \text{ is in } L^1. \tag{26}$$

For the rest of the proof, we will consider that Δ_1 is acting on $C([-2\tau, 0])$ and the notation y_t will correspond to the translation over $[-2\tau, 0]$.

In terms of Δ_1, the equation reads as

$$\frac{dy}{dt} = \Delta_1(t, y_t) + \tilde{\Lambda}(t) \cdot y(t) + h(t, y_t). \tag{27}$$

Second Change of Variables

Variable 3. $z(t) = \exp(-\int^t \tilde{\Lambda}(s)\, ds) \cdot y(t)$. Using (27), we obtain the following equation in z:

$$\frac{dz}{dt} = \exp\left(-\int^t \tilde{\Lambda}(s)\, ds\right) \cdot \Delta_1\left(t, \exp\left(\int^{t+} \tilde{\Lambda}(s)\, ds\right) \cdot z_t\right)$$
$$+ \exp\left(-\int^t \tilde{\Lambda}(s)\, ds\right) \cdot h\left(t, \exp\left(\int^{t+\cdot} \tilde{\Lambda}(s)\, ds\right) \cdot z_t\right). \tag{28}$$

Because Δ_1 is diagonal, the first term in the right side of (28) reduces to $\Delta_2(t, z_t)$, where

Notation 7. $\Delta_2(t, z_t) = \Delta_1(t, \exp(\int_t^{t+\cdot} \tilde{\Lambda}(s)\, ds) \cdot z_t)$.

From (26) we see that

$$\|\Delta_2(t, \cdot)\| \text{ is in } L^1. \tag{29}$$

Notation 8. $h_2(t, \varphi) = \exp(-\int^t \tilde{\Lambda}(s)\, ds) \cdot h(t, \exp(\int^{t+\cdot} \tilde{\Lambda}(s)\, ds) \cdot \varphi)$.

Because $\tilde{\Lambda}(s)$ is in L^2, we have $\|\int^t \tilde{\Lambda}(s)\, ds\| = O(\sqrt{t})$, and therefore $|h_2(t, \varphi)| \leqslant \exp(C_1 \cdot \sqrt{t}) \cdot C_2 \cdot e^{\eta t} \cdot l(t) \exp(C_2 \sqrt{t}) \cdot |\varphi|$. So $\|h_2(t, \cdot)\| = l(t) \cdot O(e^{(\eta/2)t})$, which implies in particular that

$$\|h_2(t, \cdot)\| \text{ is in } L^1. \tag{30}$$

In terms of Δ_2, h_2, Eq. (28) reads as

$$\frac{dz}{dt} = \Delta_2(t, z_t) + h_2(t, z_t), \tag{31}$$

which, in view of (29) and (30), verifies the assumption (H_4) of Proposition 1. So, from Proposition 1, it follows first that $z(t) = c + o(1)$, and, coming back to x, we get the formula (20) of Theorem 1.

The continuity of the limiting functional is also a consequence of Proposition 1 and Remark 1, as well as the surjectivity. But, surjectivity holds with respect to the set of data for Eq. (31), which is $C([-2\tau, 0], \mathbb{R}^n)$.

The solutions of (3) constitute only a subset of this set. We must then show that there is still surjectivity with respect to the solutions of (3) or, equivalently, to the solutions of (21). The reason for this is to consider special data for (21) (as we did when considering the surjectivity for Corollary 1). We take $y_{t_0} = \exp(\int^{t_0+} \tilde{A}(s)\, ds) \cdot c$, which, by the transformation induced in Variable 3, leads to a set of data for (31) of the form

$$z/[t_0 - \tau, t_0] = c \qquad \text{and} \qquad z/[t_0, t_0 + \tau] = \exp\left(-\int^{\cdot} \tilde{A}\right) \cdot y/[t_0, t_0 + \tau].$$

Denote by $\tilde{c} = z/[t_0 - \tau, t_0 + \tau]$. If t_0 is large enough we can prove that $\tilde{c}$ will be close to c.

On the other hand, we know (see Remark 1) that, if we denote by $z(t_0, \cdot)(\infty) = \lim_{t \to \infty} z(t_0, \cdot)(t)$, this functional tends (uniformly with respect to the bounded sets) to δ_0, so that (1) $z(t_0, \cdot)(\infty)$ is uniformly bounded (in t_0) and (2) $z(t_0, c)(\infty) = c + \varepsilon(t_0) \cdot c$, with $\|\varepsilon(t_0)\| \to 0$, as $t_0 \to +\infty$. Using (1) and (2), we see that for t_0 large enough

$$|z(t_0 + \tau, \tilde{c})(\infty)| \geqslant |c| - |\varepsilon(\tau_0 + \tau)| \cdot |c| - M \cdot |c - \tilde{c}| > 0, \qquad \text{for } |c| = 1.$$

This implies the surjectivity with respect to the restricted space, and, so, the surjectivity of the limiting functional associated with (3) which completes the proof of Theorem 1.

3. The General Case

We now assume only (H_1) and (H_2). Using Proposition 2 and its corollary, we will prove the following:

THEOREM 2. *Assume* (H_1) *and* (H_2). *Then, there exists a family of matrix-valued functionals* $\varepsilon(t, t_0, \cdot) = (\varepsilon_{ij}(t, t_0, \cdot))_{1 \leqslant i,j \leqslant n}$, $t \geqslant t_0$ *such that*

(i) $\varepsilon_{ij} = 0$, *for* $j \geqslant i$;

(ii) $\varepsilon_{ij}(t, t_0, \cdot)$ *is a bounded functional on* $C([t_0 - 2\tau, t], \mathbb{R}^n)$;

(iii) $\|\varepsilon_{ij}(t, t_0, \cdot)\| \to 0$ *as* $t \to +\infty$ (*or, as* $t_0 \to +\infty$);

(iv) $t \to \|\varepsilon_{ij}(t, t_0, \cdot)\|$ *is in* $L^2(t_0, +\infty)$;

and for every solution x *of* (3) *on* $[t_0, +\infty)$, *there exists a constant* c *in* $\mathbb{R}^n$ *such that*

$$x(t) = \delta_t + \varepsilon(t, t_0, \cdot))\left[\exp\left(\int^t A(s)\, ds\right)\right] \cdot (c + o(1)), \qquad t \geqslant t_0 + 2\tau, \quad (32)$$

where δ_t *denotes the Dirac distribution* (*or, the evaluation map*) *at* t, *that is* $\delta_t(\varphi) = \varphi(t)$; $A(t)$ *is as in Theorem* 1. *The functional* $x_{t_0} \to c$ *is continuous and surjective if* t_0 *is large enough.*

Remark 5. While the formula (32) follows naturally from the proof of the theorem, it may appear strange at first. We can express it in a more explicit form which refers us to formula (7). For that, consider for simplicity that $t_0 = 0$. Denote by $G(t) = \varepsilon(t, 0, \cdot) \cdot G(t)$ is a functional on $C([-2\tau, \tau], \mathbb{R}^n)$ such that $\|G(t)\| \to 0$ as $t \to +\infty$. Now, the theorem says that for each x solution of (3), there exists a constant c and a function $\eta(t, \cdot) \in C([-2\tau, t], \mathbb{R}^n)$, $\|\eta(t, \cdot)\| \to 0$ as $t \to +\infty$, such that

$$x(t) = \exp\left[\int_0^t A(s)\,ds\right] \cdot (c + \eta(t, t))$$
$$+ G(t) \cdot \left\{\exp\left[\int_0^t A(s)\,ds\right] \cdot \eta(t, \cdot)\right\}. \tag{33}$$

Proof. The first step of the proof is to write (3) under the form of (9). We decompose the variable $x = (x_1, z)$, $x_1 \in \mathbb{R}$, $z \in \mathbb{R}^{n-1}$ and change the variables.

Variable 4. $u(t) = \exp(-\int^t \lambda_1(s)\,ds) \cdot x_1(t)$; $y(t) = \exp(-\int^t \lambda_1(s)\,ds) \cdot z(t)$ (where $\lambda_1(t) = A_{11}(t)$). (u, y) satisfies the equation

$$\begin{aligned} \frac{du}{dt} &= \alpha(t, u_t) + \beta(t, y_t), \\ \frac{dy}{dt} &= \gamma(t, u_t) + \delta(t, y_t). \end{aligned} \tag{34}$$

LEMMA 2. $\alpha(t, \cdot)$ (resp. $\beta(t, \cdot)$) *are bounded linear functionals on* $C([-2\tau, 0], \mathbb{R})$ (resp. $C([-2\tau, 0], \mathbb{R}^{n-1})$). *Moreover* $\|\alpha(t, \cdot)\|$ *is in* L^1 *and* $\|\beta(t, \cdot)\|$ *is in* L^2.

Let us prove these results.

After changing the variables (x, z) into (u, y), the equation in u takes the form

$$\frac{du}{dt} = L_{11}\left(t, \exp\left(\int_t^{t+} \lambda_1(s)\,ds \cdot u_t\right) - L_{11}(t, \exp(\lambda_1 \cdot) \cdot u(t))\right.$$
$$+ \sum_{j=2}^{n} L_{1,j}\left(t, \exp\left(\int_t^{t+} \lambda_1(s)\,ds\right) \cdot (y_j)_t\right. \tag{35}$$

The first group of terms in (35) corresponds to $\alpha(t, u_t)$ in (34) while the other which depends only on y corresponds to $\beta(t, y_t)$. For this last term, we can see that it satisfies the properties stated in Lemma 2, that is, $\|\beta(t, \cdot)\|$ is in L^2.

We only have to consider the terms in u. We observe that

$$\int_t^{t+s} \lambda_1(\theta)\, d\theta = \lambda_1 s + \int_t^{t+s} L_{11}(\theta, e^{\lambda_1})\, d\theta,$$

and $t \to \int_t^{t+s} L_{11}(\theta, e^{\lambda_1})\, d\theta$ is in L^2 and L^∞, uniformly with respect to $s \in [-\tau, 0]$. We will denote this fact by writing this function as $O(L^2 \cap L^\infty)$. So $\exp \int_t^{t+} \lambda_1(s)\, ds = \exp(\lambda_1 t) \cdot (1 + O(L^2 \cap L^\infty))$. Then the first term in (35) can be written as

$$L_{11}(t, \exp(\lambda_1 \cdot) \cdot u_t) + \alpha_1(t, u_t),$$

where $\alpha_1(t, \cdot)$ verifies the desired property that $\|\alpha_1(t, \cdot)\|$ is in L^1 (and, in fact, $\|\alpha_1(t, \cdot)\|$ is also in L^2).

So, we have to concentrate on $L_{11}(t, \exp(\lambda_1 \cdot)(u_t - u(t)))$, and we will express $u_t - u(t)$ in terms of the derivative of u, and, then of $u/[t-2\tau, t]$. Using (35), we have $u_t - u(t) = \int_t^{t+} du/ds \cdot ds$, so that

$$\begin{aligned}
&L_{11}(t, \exp(\lambda_1 \cdot) \cdot (u_t - u(t)) \\
&\quad = L_{11}(t, \exp(\lambda_1 \cdot) \cdot \int_t^{t+} L_{11}(s, \exp(\lambda_1 \cdot) \cdot (u_s - u(s)))\, ds + L_{11}(t, \exp(\lambda_1 \cdot) \\
&\quad \cdot \int_t^{t+} L_{11}(s, O(L^2 \cap L^\infty) \cdot u_s)\, ds + \sum_{i=2}^{n} L_{11}(t, \exp(\lambda_1 \cdot) \\
&\quad \cdot \int_t^{t+} L_{1,i} \int_t^{t+} L_{1,i}\left(s, \exp\left(\int_s^{s+} \lambda_1\right) \cdot (y_i)_s\right) ds\right).
\end{aligned}$$

Once again using arguments on products of L^p, L^q functions and the fact that the operator $\int_t^{t+}$ sends L^p into $L^p \cap L^\infty$ for each $p \geqslant 1$, we see that the first operator in the right side has its time-dependent norm in L^1, as well as the second one, and the third one is in $L^1 \cap L^2$. Thus, the proof of Lemma 2 is complete.

We turn now to the consideration of $\gamma(t, u_t)$ and $\delta(t, y_t)$ in (34).

LEMMA 3. *$\|\gamma(t, \cdot)\|$ is in L^2, and the equation*

$$\frac{dy}{dt} = \delta(t, y_t) \textit{ is exponentially stable.} \tag{36}$$

The proof of this lemma is straightforward.

We only have to express γ and δ in terms of the original equation. Immediately from the change of variables, we get

$$\gamma(t, \varphi) = \left(L_{1,j}\left(t, \exp\left(\int_t^{t+} \lambda_1(s)\, ds\right)\right) \cdot \varphi\right), \qquad j = 2, ..., n$$

so that $\|\gamma(t, \cdot)\|$ is in L^2 and

$$\delta(t, \varphi) = (A'\lambda_1(t)\, I')\, \varphi(0) + L'\left(t, \exp\left(\int_t^{t+} \lambda_1(s)\, ds\right) \cdot \varphi\right)$$

(where we denote by the prime the restriction of operators to $\mathbb{R}^{n-1}$). $A' - \lambda_1(t)\, I'$ is diagonal with ith elements

$$\lambda_i - \lambda_1 - L_{11}(t, e^{\lambda_1}), \qquad i = 2, ..., n,$$
$$\lambda_i - \lambda_1 < 0,$$

and $L_{11}(t, e^{\lambda_1})$ is an L^2-perturbation, as well as the last term in the right side of $\delta(t, \varphi)$. This ensures the exponential stability of (36).

The conclusion from these two lemmas is that (34) verifies the conditions of Proposition 2, and, more specifically, its corollary. Therefore the conclusions of Corollary 1 hold:

$$\lim_{t \to +\infty} y(t) = 0 \qquad \text{and} \qquad \lim_{t \to +\infty} u(t) \text{ exists.}$$

This limit is a continuous functional of the data of the original equation (3). Concerning the surjectivity, the same problem as in Theorem 1 arises. Surjectivity holds, for t_0 large enough, with respect to the data in $C([-2\tau, 0], \mathbb{R}^n)$. A similar argument proves that it holds also with respect to $C([-\tau, 0], \mathbb{R}^n)$.

We come back now to the original variables (x_1, z). We will denote by $V(t, s)$ (as in Notation 1) the solution operator for Eq. (36). We get a first expression for (x_1, z),

$$x_1(t) = \exp\left(\int^t \lambda_1(s)\, ds\right) \cdot [c_1 + o(1)], \tag{37}$$

$$z(t) = \exp\left(\int^t \lambda_1(s)\, ds\right) \cdot \left[V(t, t_0) \cdot y_{t_0}(0) + \int_{t_0}^t (V(t, s)\, Y_0)(0) \cdot \gamma(s, u_s)\, ds\right] \tag{38}$$

(where $c_1 = \lim_{t \to +\infty} u(t)$).

The right side of (38) can be decomposed into a sum: $z = z^{(1)}(t) + z^{(2)}(t)$.

Variable 5. $z^{(2)}(t) = \exp(\int^t \lambda_1(s)\, ds) \int_{t_0}^t (V(t, s) \cdot Y_0)(0)\, \gamma(s, u_s)\, ds$.

Notation 9. For $t \geqslant t_0$, we denote by $\varepsilon_1(t, t_0, \cdot)$ the following functional from $C([t_0 - 2\tau, t], \mathbb{R})$ into $\mathbb{R}^n$: $\varepsilon_1 = (\varepsilon_{1j})$ where $\varepsilon_{11} = 0$ and

$$\varepsilon_{1j}(t, t_0, \varphi) = \int_{t_0}^t (V(t, s) \cdot Y_0)_j(0)\, \gamma(s, \varphi_s)\, ds, \qquad j = 2, ..., n.$$

LEMMA 4. *ε_{1j}, $j = 1, \ldots, n$, verifies the properties* (i)–(iv) *stated in Theorem* 2.

Proof. (i) follows from Notation 9; (ii), (iv) are direct consequences of the estimate

$$\|\varepsilon_1(t, t_0, \cdot)\| \leqslant K \int_{t_0}^{t} \exp -\alpha(t-s).\ \|\gamma(s, \cdot)\|\, ds, \tag{39}$$

which is due to (12), and the fact that $\|\gamma(s, \cdot)\|$ is in L^2 (Lemma 3). In terms of ε_1, $z^{(2)}$ is given by

$$z^{(2)}(t) = \varepsilon_1(t, t_0, \cdot)\left(\exp \int^{t} \lambda_1(s)\, ds\right) \cdot [c_1 + o(1)] \tag{40}$$

and

$$z^{(1)}(t) = \exp\left(\int^{t} \lambda_1(s)\, ds\right) (V(t, t_0) \cdot y_{t_0})(0)). \tag{41}$$

This means that $z^{(1)}$ is a solution of

$$\frac{dz}{dt} = A' z(t) + L'(t, z_t), \tag{42}$$

where, as in Lemma 3, A', L' correspond to the restriction of A, L to $\mathbb{R}^{n-1}$.

We are now ready to complete the proof of Theorem 2. After the first step, the problem has been reduced to a problem of the same type with one dimension less. This suggests an iterative procedure and the formulas (37) and (40) give an indication of what will be the general expression. In fact, assuming that we can find an expression like (32) for (42), we will then have

$$z^{(1)}(t) = (\delta_t + \varepsilon'(t, t_0, \cdot)) \exp\left(\int^{t} A'(s)\, ds\right) \cdot [c' + o(1)], \qquad t \geqslant t_0 + 2\tau. \tag{43}$$

Combining (37), (40), and (43) we obtain (32) for $x = (x_1, z)$. The proof of Theorem 2 is complete.

In conclusion, we emphasize a very striking difference between the results obtained here (which are more or less those conjectured in [9]) and the asymptotic integration results in the ordinary case. In [13], Hartman and Wintner recalled a first theorem in this field due to O. Perron [15]: assuming that $A(t)$ (in Eq. (5)) tends to 0 at infinity and A has n distinct eigenvalues, $(\lambda_i)_{i=1,\ldots,n}$; (5) has n independent solutions (x_i), with

$$\|x_i(t)\| \sim e^{\lambda_i t}, \qquad t \to +\infty.$$

They obtained extensions of this result, still with the existence of n different exponential growths.

In [11] Harris and Lutz consider also an ordinary equation and show, under certain conditions, the existence of a fundamental solution of the form

$$Y(t) = [I + o(1)] \exp \int^t \Lambda(s)\, ds. \tag{44}$$

This formula also implies the existence of n distinct exponential growths.

In contrast to that, even in the triangular case, the results we get do not allow us to separate different asymptotic exponential growths. The best interpretation we can make in this direction is to say that each solution can be written as a combination of functions with distinct growths. The delay seems to prevent formulas such as (44). There are some cases however where we can at least find solutions with distinct exponential growths, in particular, the case where the right side of (3) is "small enough." We will develop this assertion now.

We know that under an appropriate smallness condition (3) has an n dimensional subspace of "special" solutions, defined on $\mathbb{R}$, uniquely determined by their value at a point [16, 7, 8, 1, 2]. We established in [2] that these solutions are associated to an ordinary differential equation.

The "smallness condition" is that, if we denote $M(t, \varphi) = \Lambda\varphi(0) + L(t, \varphi)$, we have $\|M(t, \cdot)\| \leqslant K$ and $K\tau e < 1$. Under this condition special solutions exist [1, 6, 7] and associated to them is a function $G(s, \theta)$ from $\mathbb{R} \times [-\tau, 0]$ into the set of $n \times n$ matrices, so that for every special solution, $x(s+\theta) = G(s, \theta) \cdot x(s)$. And now if we define $m(t, x) = M(t, G(t, \cdot)\, x)$, the special solutions x are the solutions of the ordinary equation $dx/dt = m(t, x(t))$.

Because $|G(s, \theta)| \leqslant C$ ([2]) and $G(s, 0) = I$. we have $m(t, x) = \Lambda x + n(t, x)$, with $\|n(t, \cdot)\|$ in L^2. The equation in m satisfies the conditions for asymptotic integration given in [11, 13]. Therefore, it has a fundamental solution of the type (44) and, in particular, n solutions with distinct exponential growths.

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